On Lipschitz continuity of the iterated function system in a stochastic optimal growth model

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ABSTRACT

This paper provides qualitative properties of the iterated function system (IFS) generated by the optimal policy function for a class of stochastic one-sector optimal growth models. We obtain, explicitly in terms of the primitives of the model (i) a compact interval (not including the zero stock) in which the support of the invariant distribution of output must lie, and (ii) a Lipschitz property of the iterated function system on this interval. As applications, we are able to present parameter configurations under which (a) the support of the invariant distribution of the IFS is a generalized Cantor set, and (b) the invariant distribution is singular.

1. Introduction

This paper constitutes a contribution in the study of qualitative properties of the iterated function system (IFS) generated by the optimal policy function for a class of stochastic one-sector optimal growth models initially proposed by Brock and Mirman (1972) (see also Mirman and Zilcha, 1975). As an application, we investigate the nature of the invariant distribution which represents the stochastic steady state of an optimal growth model under uncertainty, and the geometric properties of its support.

The main qualitative properties of the iterated function system derived in this paper can be described as follows. We obtain, explicitly in terms of the primitives of the model (i) a compact interval (not including the zero stock) in which the support of the invariant distribution of output must lie, and (ii) a Lipschitz property of the iterated function system on this interval. We elaborate below on each result, and relate them to what is available in the literature.

We try to keep our framework comparable to that of Brock and Mirman (1972), but we deal only with the case in which the random production shock is multiplicative and can take on a finite number of values. Our result (i), as well as its method

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of proof (see Proposition 2) provides justification for the choice of this framework. It would be of interest to see if our result can be obtained in somewhat more general frameworks.

The Brock–Mirman framework is more general as the random shock is allowed to take values in an interval $[\alpha, \beta]$, where $0 < \alpha < \beta < \infty$. However, with that set-up, and without any further restrictions, the support of the invariant distribution cannot be shown to belong to a compact interval bounded away from zero; for a concrete counter-example, see Mirman and Zilcha (1976, pp. 124–126). A sufficient condition to ensure the result is that the probability of the random shock taking on the value α be positive. However, this condition rules out the case in which the random shock has a density with support $[\alpha, \beta]$; this point has already been made by Mirman and Zilcha (1977, p. 389 and footnote 1).

In a recent paper, Chatterjee and Shukayev (2008) allow for the general framework of Brock–Mirman, and show that if the utility function is bounded (below), then given any initial output level, $y > 0$, there exists $y' \in (0, y)$, such that the Markov process generated by the IFS, which starts from y will stay in (y', ∞) for all time. In other words, the possibility of a map of the IFS lying entirely below the $45°$ line (for all positive output levels), for some realization of the random shock, can be ruled out. This, however, still does not ensure that there is a compact interval, that excludes the zero stock, in which the support of the invariant distribution of output must lie. Further, given $y > 0$, Chatterjee and Shukayev (2008) do not provide a lower bound y', dependent on y and the primitives of the model.

Our result (ii) provides a useful qualitative property of the IFS generated by the optimal policy function for a class of stochastic one-sector optimal growth models. Its principal merit is that the Lipschitz constants (corresponding to the various maps of the IFS) can be expressed explicitly in terms of the primitives of the model.[1] This allows one to readily see (for instance) the connection between the standard measure of relative risk aversion of the representative agent and the Lipschitz property (see Theorem 1, and the remarks that follow its statement). Since there is a well-developed theory for iterated function systems in which the component functions are contractions, our result also permits us to obtain restrictions on the parameters of our models under which that theory can be invoked.

We have not been able to find properties similar to our result (ii) in the literature. Apart from monotonicity and continuity of the component maps of the IFS, very few results of a general nature are known about the IFS generated by the optimal policy function in stochastic one-sector optimal growth models.

We apply our results to study the nature of the unique invariant distribution in the special case in which the random shock takes on only two values. The first step along this direction was taken by Mitra et al. (2004), who investigated how the nature of the invariant distribution and the geometry of its support depend on the exponent of a Cobb-Douglas production function, and on the probability of a random shock affecting multiplicatively such production, when the utility of the representative agent is logarithmic. It was shown that the invariant distribution can exhibit a sufficiently rich variety of features in this simple setting. However, unlike this special production-utility specification, the difficulty in studying features of the invariant distribution in general is that the maps of the IFS are derived from optimal consumption and investment functions, and one cannot typically solve for these functions explicitly in terms of the primitives.

Given this fact, the hope is that qualitative properties of the IFS (which can be checked given the primitives) can provide the basis for qualitative statements about the invariant distribution, and this is the line of research that has been pursued in Mitra and Privileggi (2004, 2006). The former paper tackles the case of iso-elastic utility (and Cobb-Douglas production), and the latter the more general case of production and utility represented by increasing concave twice continuously differentiable functions satisfying the standard assumptions of neoclassical discounted optimal growth models. It has been shown in these papers that when the maps of the IFS are strictly monotone and contractive, and a "no-overlap" property holds, then the support of the invariant distribution is a generalized Cantor set.

However, while sufficient conditions for the no-overlap property were established in terms of the parameters of the model, the assumption of an IFS with contractive maps was made exogenously. The main result of this paper (Theorem 1) allows us to remove this shortcoming. We are now able to present sufficient conditions, in terms of the primitives of the model, for the maps of the IFS to be contractions, and satisfy the no-overlap property (Proposition 6).

Here is a brief outline of the paper. In Section 2, after stating the model, we recall some basic properties of the optimal policy function necessary to characterize the support of the invariant distribution of the IFS generated by such a policy. The central contribution of this paper is contained in Section 3, where we provide conditions under which the maps of the IFS are Lipschitz, with Lipschitz constants which can be directly computed in terms of the parameters of the model.

Section 4 is devoted to the application of the results of Section 3. Section 4.1 makes precise what we mean by a generalized topological Cantor set as the attractor of an IFS with nonlinear maps. It provides parameter restrictions under which the maps of the IFS are contractions, and they satisfy the no-overlap property, thereby ensuring that its invariant distribution is supported on a generalized topological Cantor set. Section 4.2 identifies parameter configurations under which the maps of the IFS are "sufficiently contracting" to ensure that the attractor of the IFS has Lebesgue measure zero, and thus the invariant distribution is necessarily singular. Section 5 reports some concluding remarks, while most of the proofs are gathered in Appendix A.

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[1] This aspect is related to the method of proof, which might also be useful in other contexts.

2. The framework

We consider a special case of the standard model of optimal growth under uncertainty as presented in Brock and Mirman (1972) and Mirman and Zilcha (1975). Specifically, the production function is one in which the shocks are multiplicative, so there is a function, $h : \mathbb{R}_+ \to \mathbb{R}_+$, such that $f(x, r) = rh(x)$ for $(x, r) \in \mathbb{R}_+ \times Z$. The set Z of values of the random variable, r, is $\{r_1, \ldots, r_n\}$, where $0 < r_1 < \cdots < r_n$. We interpret r_i [for $i \in I = \{1, \ldots, n\}$] as a production shock which occurs with probability $p_i \in (0, 1)$, where $\sum_{i=1}^{n} p_i = 1$.

Both the production function, h, and the utility function, u, defined on $\mathbb{R}_+$, are C^2 functions on $\mathbb{R}_{++}$ satisfying the standard assumptions as in Brock and Mirman (1972), namely:

$$h(0) = 0, \qquad h'(\cdot) > 0, \qquad h''(\cdot) < 0, \qquad \lim_{x \to 0^+} h'(x) = +\infty, \qquad \lim_{x \to +\infty} h'(x) = 0, \tag{1}$$

$$u'(\cdot) > 0, \qquad u''(\cdot) < 0, \qquad \lim_{x \to 0^+} u'(x) = +\infty. \tag{2}$$

Under (1), there is a number $k > 0$ such that $r_n h(k) = k$, $r_n h(x) > x$ for all $0 < x < k$ and $r_n h(x) < x$ for all $x > k$. Since $0 < r_1 < \cdots < r_n$, the closed interval $[0, k]$ can be taken as the state space for our model. Thus, the "primitives" of our model are the functions h and u, parameters $\{r_1, \ldots, r_n\}$, $\{p_1, \ldots, p_n\}$, a discount factor δ belonging to $(0, 1)$, and the positive value k which depends[2] on h.

One can apply the standard theory of stochastic dynamic programming to obtain an (optimal) value function, $V : \mathbb{R}_+ \to \mathbb{R}_+$ and two (optimal) policy functions, $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$, which we will interpret as the consumption and the investment functions respectively. That is, given any output level, $y \geq 0$, the optimal consumption out of this output is given by $g(y)$, while the optimal input choice (for production in the next period) is then $\gamma(y) = [y - g(y)]$. We denote $h[\gamma(y)]$ by $G(y)$; the output obtained in the next period, when r takes the value r_i (for $i \in I$), is then given by $r_i G(y)$. This convenient simplification comes from the multiplicative form of the production shock.

Following Brock and Mirman (1972) and Mirman and Zilcha (1975), one can establish several useful properties of the value and policy functions. We summarize these results (without proofs) in the following Proposition.

Proposition 1. *The value function, V, and the policy function, g, satisfy the following properties*:

 (i) *V is concave on* $\mathbb{R}_+$, *and continuous on* $\mathbb{R}_{++}$;
 (ii) *g is continuous on* $\mathbb{R}_+$ *and* $0 < g(y) < y$ *for* $y > 0$;
(iii) *$g(y)$ and $\gamma(y)$ are both strictly increasing in y on* $\mathbb{R}_+$;
(iv) *V is continuously differentiable on* $\mathbb{R}_{++}$, *and* $V'(y) = u'[g(y)]$ *for* $y > 0$;
 (v) *for* $y > 0$, *we have*

$$u'[g(y)] = \delta \mathbb{E}\{V'[rG(y)]rh'[\gamma(y)]\}; \tag{3}$$

(vi) *for* $y > 0$, *we have*

$$u'[g(y)] = \delta \mathbb{E}\{u'[g(rG(y))]rh'[\gamma(y)]\}. \tag{4}$$

The optimal policy function leads to the stochastic process:

$$y_{t+1} = r_{t+1} G(y_t) \quad \text{for } t \geq 0, \tag{5}$$

where $r_{t+1} \in Z = \{r_1, \ldots, r_n\}$. Alternately, one might say that the optimal policy function leads to an *iterated function system* (IFS) $\{G_1, \ldots, G_n; p_1, \ldots, p_n\}$, where G_i is defined for $i \in I$ by $G_i(y) = r_i G(y)$ for $y \in \mathbb{R}_+$. It is known (Brock and Mirman, 1972) that there is a unique invariant distribution, μ, of the Markov process described by (5), and the distribution of optimal output at date t, call it μ_t, converges weakly to μ.[3] We are principally interested in the nature of this distribution μ and in the geometric properties of its support. The distribution function corresponding to μ is denoted by F.

It can be checked that the functions G_i have positive fixed points, and all the fixed points are less than k. Denote by a the largest fixed point of G_1, and by b the smallest fixed point of G_n. Following Brock and Mirman (1972), one can establish that $a < b$. The interval $[a, b]$ is an invariant stable set of the stochastic process (5). In particular, the support of F is contained in $[a, b]$. Consequently, in studying the nature of F, it is enough to concentrate on the stochastic process (5), with initial output $y \in [a, b]$. Equivalently, one need only to study the IFS $\{G_1, \ldots, G_n; p_1, \ldots, p_n\}$ on the interval $X = [a, b]$.

However, the interval $X = [a, b]$ is not known to us (in terms of the primitives); nor are the functions $\{G_1, \ldots, G_n\}$. They depend on the optimal policy functions, which typically cannot be solved explicitly in terms of the primitives. Consequently, in order to study the behavior of the iterated function system, we need to obtain some qualitative properties of the iterated function system, which can be explicitly checked, given the primitives.

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[2] Denoting $[h(x)/x]$ by $H(x)$ for $x > 0$, we note that H is a monotonic decreasing function, and has an inverse. Then, $k = H^{-1}(1/r_n)$. In the case where h is of the Cobb-Douglas form: $h(x) = x^{1-\alpha}/(1 - \alpha)$ for $x \geq 0$, where $\alpha \in (0, 1)$, it can be checked that $k = [r_n/(1 - \alpha)]^{(1/\alpha)}$.

[3] For an alternate and simpler approach to this result, see Bhattacharya and Majumdar (2001).

3. Lipschitz continuity of the iterated function system

In this section, we present two of the main qualitative results of this paper. We obtain, explicitly in terms of the primitives of the model (i) a compact interval (not including the zero stock) in which the support of the invariant distribution of output must lie, and (ii) a Lipschitz property of the iterated function system on this interval. To elaborate, the latter property establishes that each function G_i (for $i \in I$) is Lipschitz continuous on this interval, with a Lipschitz constant expressed in terms of the primitives of the model.

3.1. A bound on the support of the invariant distribution

The following proposition establishes a positive lower bound for every fixed point a' of the (lowest) map G_1; this lower bound (denoted by θ) is obtained explicitly in terms of the primitives of the model. Thus, $[\theta, k]$ provides a compact interval (not including the zero stock) in which the support of the invariant distribution of output must lie. The result is crucial to all the subsequent analysis.

Note that we cannot find this lower bound on the fixed point of the lowest map if the "smallest" value of the random shock does not occur with positive probability. This can happen when the random shock can take on an infinite number of values, and in particular when the distribution of the random shock has a density. This justifies our choice of a setting in which the random shock can assume a finite number of values with positive probabilities.

Since the above observation can be appreciated by noting how the lower bound is obtained, we provide the proof of the proposition in the text itself (instead of in Appendix A).

For any function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $f'(\cdot)$ is strictly monotone on $\mathbb{R}_+$ we denote by $(f')^{-1}(\cdot)$ the inverse function of its derivative on $f(\mathbb{R}_+)$.

Proposition 2. *Define*:

$$\eta = (h')^{-1} \left(\frac{1}{\delta p_1 r_1} \right), \tag{6}$$

$$\theta = r_1 h(\eta). \tag{7}$$

 (i) *For every positive fixed point a' of the map G_1, we have*:

$$a' > \theta. \tag{8}$$

 (ii) *For every $y \in (0, \theta]$, we have*:

$$G_1(y) > y. \tag{9}$$

(iii) *For every positive fixed point a' of the map G_1, we have*:

$$\gamma(a') > \gamma(\theta) > \eta, \tag{10}$$

$$h'[\gamma(a')] < h'[\gamma(\theta)] < h'(\eta) = \frac{1}{\delta p_1 r_1}. \tag{11}$$

Proof.

(i) Using (4) for $y = a'$, where a' is an arbitrary positive fixed point of G_1, we get:

$$\begin{aligned} u'[g(a')] \quad &= \delta \sum_{i=1}^{n} p_i u'\{g[r_i G(a')]\} r_i h'[\gamma(a')] \\ &> \delta p_1 u'\{g[r_1 G(a')]\} r_1 h'[\gamma(a')] \\ &= \delta p_1 u'[g(a')] r_1 h'[\gamma(a')], \end{aligned}$$

the last line following from the fact that $r_1 G(a') = a'$. Then, since h' is a strictly decreasing function, we obtain:

$$\gamma(a') > \eta. \tag{12}$$

Using (12) and the definition of a', we obtain:

$$a' = r_1 h[\gamma(a')] > r_1 h(\eta) = \theta,$$

which is (8).

(ii) Suppose, on the contrary, there is some $y \in (0, \theta]$, such that (9) does not hold. Since all positive fixed points of G_1 exceed θ, we must have $G_1(y) < y$ for all $y \in (0, \theta]$. Then, for all $y \in (0, \theta]$, we can use (4) to obtain:

$$
\begin{aligned}
u'[g(y)] \quad &= \delta \sum_{i=1}^{n} p_i u'\{g[r_i G(y)]\} r_i h'[\gamma(y)] \\
&> \delta p_1 u'\{g[r_1 G(y)]\} r_1 h'[\gamma(y)] \\
&> \delta p_1 u'[g(y)] r_1 h'[\gamma(y)] \\
&> \delta p_1 u'[g(y)] r_1 h'(y),
\end{aligned}
$$

where the last but one line holds as $G_1(y) = r_1 G(y) < y$. Hence,

$$
\delta p_1 r_1 h'(y) < 1 \quad \text{for all } y \in (0, \theta],
$$

but this leads to a contradiction, since the Inada condition entails $h'(y) \to \infty$ as $y \to 0$.

(iii) Using (9) and (7), we have:

$$
h[\gamma(\theta)] = G(\theta) > \frac{\theta}{r_1} = h(\eta). \tag{13}
$$

Combining (8) and (13) yields (10). Now, (11) follows from (10), (6) and the fact that h' is a strictly decreasing function. □

A useful observation that can be obtained from the above result is that the interval $Y = [\theta, k]$ is an invariant stable set of the stochastic process (5). Since G_i is an increasing function for each $i \in I$, Proposition 2(ii) yields for all $y \in Y = [\theta, k]$, and all $i \in I$,

$$
\theta < G_1(\theta) \leq G_i(y) \leq G_n(y) = r_n h[\gamma(y)] \leq r_n h(y) \leq r_n h(k) = k,
$$

so that each G_i maps Y into Y.

3.2. A Lipschitz property

We will now show that the iterated function system $\{G_1, \ldots, G_n; p_1, \ldots, p_n\}$ on the interval $Y = [\theta, k]$ has the Lipschitz property; that is, the maps G_i (for $i \in I$) are Lipschitz continuous on Y. It is sufficient for this purpose to show that G is Lipschitz continuous on Y.

The Lipschitz continuity of policy functions in non-stochastic models has been studied by Montrucchio (1987). Given the structure of our model, we are able to take a fairly direct approach. The Lipschitz constant relative to function G_i will depend only on the parameters of the model. In obtaining this result, use is made of the facts that (i) the deterministic component, h, of the production function exhibits uniformly bounded elasticity of marginal product, and (ii) the utility function u exhibits uniformly bounded elasticity of marginal utility (a measure of relative risk aversion). These facts, reported as Fact 1 and Fact 2 below, follow from our assumptions on h and u in Section 2.

Keeping this objective in mind, we first obtain a positive lower bound on the optimal propensity to consume, $[g(y)/y]$. This result is clearly also of independent interest as a property of the optimal consumption function.

The following Lemma establishes an intermediate value (which depends on the parameters of the model) separating the optimal investments at the fixed points a and b. Let $\rho = \mathbb{E}(r) = \sum_{i=1}^{n} p_i r_i$ denote the expected value of r.

Lemma 1. *Values $\gamma(a)$ and $\gamma(b)$ satisfy the following relation*:

$$
\gamma(\theta) < \gamma(a) \leq (h')^{-1}\left(\frac{1}{\delta \rho}\right) \leq \gamma(b) \leq \gamma(k). \tag{14}
$$

The proof is reported in Appendix A. It is convenient to label the intermediate value in (14) as follows:

$$
\sigma = (h')^{-1}\left(\frac{1}{\delta \rho}\right). \tag{15}
$$

Proposition 3. *Assume that*:

$$
\frac{\theta}{k} > \frac{\sigma}{r_n h(\sigma)}. \tag{16}
$$

Then, we have the following lower bound on the optimal propensity to consume:

$$\frac{g(y)}{y} > \frac{\theta}{k} - \frac{\sigma}{r_n h(\sigma)} \quad \text{for all } y \in Y. \tag{17}$$

The proof is reported in Appendix A.

Remark 1.

(i) Note that condition (16) in Proposition 3 does not depend on the utility function u, since θ [see (7)] and σ [see (15)] are both independent of u.

(ii) If $h(x)$ has the Cobb-Douglas form, that is, $h(x) = x^{1-\alpha}/(1-\alpha)$ for $x \geq 0$, where $\alpha \in (0, 1)$, then $[\sigma/h(\sigma)] = (1-\alpha)\delta\rho$. Then, (16) is equivalent to:

$$(p_1)^{(1-\alpha)/\alpha}[\delta(1-\alpha)]^{(1-2\alpha)/\alpha}\left(\frac{r_1}{r_n}\right)^{(1/\alpha)} > \frac{\rho}{r_n} \tag{18}$$

To interpret (18), let us measure risk by any continuous increasing function of the spread $(r_n - r_1)$ of the random shocks $\{r_1, \ldots, r_n\}$. Then, given any risk, there is α close enough to 1 (capital share in income small enough) for which (18) is satisfied.

To establish the main result of this subsection (the Lipschitz continuity of the function G) we will use the following properties, which hold on[4] $Y = [\theta, k]$, given our assumptions on h and u in Section 2.

Fact 1. *There exist numbers A and A′ such that $0 < A \leq A' < \infty$, and for all $x \in Y$,*

$$A \leq \mathbb{E}(x) \leq A', \tag{19}$$

where $E(x) = [-xh''(x)/h'(x)]$ for all $x > 0$. Further, there exists $A'' \in (0, 1)$ such that:

$$\frac{e(x')}{e(x)} \geq A'' \quad \text{for all } x, x' \in Y, \tag{20}$$

where $e(x) = xh'(x)/h(x)$ for all $x > 0$.

Fact 2. *There exist numbers B and B′ such that $0 < B \leq B' < \infty$, and for all $c \in Y$,*

$$B \leq R(c) \leq B', \tag{21}$$

where $R(c) = [-cu''(c)/u'(c)]$ for all $c > 0$.

Since Y is a non-empty compact set, and $E(x)$ is a continuous function on Y, it has a maximum and a minimum on this set. Denote the minimum by A and the maximum by A'. Then, $0 < A \leq A' < \infty$, since $E(x)$ is a positive function on Y; further, (19) clearly holds. Since $e(x)$ is also a continuous function on Y, it has a minimum on Y. Denote this minimum by A''. Then, since $e(x) \in (0, 1)$ for all $x > 0$, we have $A'' \in (0, 1)$, and clearly $[e(x')/e(x)] \geq A''/e(x) \geq A''$ for all $x, x' \in Y$. This establishes Fact 1.

Since Y is a non-empty compact set, and $R(c)$ is a continuous function on Y, it has a maximum and a minimum on this set. Denote the minimum by B and the maximum by B'. Then, $0 < B \leq B' < \infty$, since $R(c)$ is a positive function on Y; further, (21) clearly holds. This establishes Fact 2.

Denote the right hand side of condition (17), $[\theta/k] - [\sigma/r_n h(\sigma)]$, by m. Note that $m < 1$. For, if $m \leq 0$, then this is trivially true, while if $m > 0$, then this inequality follows from Proposition 1(ii) and Proposition 3.

Theorem 1. *Suppose that (16) holds*; *that is, $m > 0$. Define*:

$$L = \frac{\beta'(1-m)}{\delta p_1 r_1 [\beta'(1-m) + \alpha'm]}, \tag{22}$$

where $\alpha' = (A^2 A''/A')$ and $\beta' = [(B')^2/B]$, and constants A, A′, B and B′ are given in Facts 1 and 2. Then, for all $y, z \in Y$, we have:

$$|G(y) - G(z)| \leq L|y - z|. \tag{23}$$

The proof is reported in Appendix A.

Remark 2.

(i) If $h(x)$ has the Cobb-Douglas form, that is, $h(x) = x^{1-\alpha}/(1-\alpha)$ for $x \geq 0$, where $\alpha \in (0, 1)$, and $u(c)$ is of the iso-elastic type, that is, $u(c) = c^{1-\beta}/(1-\beta)$ for $c \geq 0$, where $\beta \in (0, 1)$, then it is easily seen that $A = A' = \alpha$, $A'' = 1$ and $B = B' = \beta$, and thus also $(A^2 A''/A') = \alpha$ and $[(B')^2/B] = \beta$. Therefore, in this case, the Lipschitz constant L defined in (17) turns out to be:

$$L = \frac{\beta(1-m)}{\delta p_1 r_1 \left[\beta(1-m) + \alpha m\right]} \leq \frac{\beta(1-m)}{\delta p_1 r_1 \alpha m}. \tag{24}$$

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[4] We thank an anonymous referee who helped us to improve on a previous statement of Facts 1 and 2.

(ii) Since m is independent of the utility function, (24) provides a particularly convenient form of the Lipschitz constant. Given the other parameters of the model such that $m > 0$, one can ensure that each G_i (for $i \in I$) is a contraction by choosing $\beta \in (0, 1)$ to be small enough. In particular, if β satisfies:

$$\beta < \frac{\delta p_1 r_1 \alpha m}{r_n(1 - m)}, \tag{25}$$

then each G_i (for $i \in I$) is a contraction. Since β is a measure of relative risk aversion, this tells us that low risk aversion leads to an IFS with the contraction property.

3.3. General results from the theory of IFS

In this section we briefly recall the main facts known in the standard theory of IFS (see, e.g., Hutchinson, 1981; Barnsley, 1993; Lasota and Mackey, 1994; Falconer, 1997, 2003). We slightly generalize the setting by considering any collection of continuous maps $\{H_1, \ldots, H_n\}$ defined on some compact subset X of the real line; that is, we shall study a generic IFS $\{H_1, \ldots, H_n; p_1, \ldots, p_n\}$, abstracting from the maps $\{G_1, \ldots, G_n\}$ discussed so far.

Let $X \subset \mathbb{R}$ be a compact set. Let $\mathcal{B}(X)$ denote the sigma-algebra of Borel measurable subsets of X and $\mathcal{P}(X)$ the space of probability measures on $\mathcal{B}(X)$. Recall that the *Barnsley operator* $S : \mathcal{B}(X) \to \mathcal{B}(X)$ is defined by

$$S(E) = \bigcup_{i=1}^{n} H_i(E) \quad \text{for } E \in \mathcal{B}(X), \tag{26}$$

and the *Markov operator* $M : \mathcal{P}(X) \to \mathcal{P}(X)$ is defined by

$$M\mu(B) = \sum_{i=1}^{n} p_i \mu[H_i^{-1}(B)] \quad \text{for } \mu \in \mathcal{P}(X), \text{ and } B \in \mathcal{B}(X),$$

where $H_i^{-1}(B)$ denote the inverse-image sets of the set B through the maps H_i. Operator M describes the evolution of probabilities under the stochastic process:

$$y_{t+1} = H_{z_t}(y_t), \tag{27}$$

where z_t are i.i.d. over $\{1, \ldots, n\}$ with distribution $\{p_1, \ldots, p_n\}$ for all $t \geq 0$. We shall denote the iterates of such operators by $S^t(E) = S[S^{t-1}(E)]$ and $M^t(\mu) = M[M^{t-1}(\mu)]$ for all $t \geq 1$, with $S^0(E) = E$ and $M^0(\mu) = \mu$.

In the next proposition are reported (without proof) the main results regarding the attractor and the unique invariant distribution of the IFS $\{H_1, \ldots, H_n; p_1, \ldots, p_n\}$ on the space $X \subset \mathbb{R}$ induced by the stochastic process (27) when the maps H_i are *contractions*.

Proposition 4. *If constants ℓ_i exist such that $0 < \ell_i < 1$ and $|H_i(y) - H_i(z)| \leq \ell_i|y - z|$ for all $y, z \in X$, $i \in I$, then the IFS $\{H_1, \ldots, H_n; p_1, \ldots, p_n\}$ satisfies the following properties*:

(i) *there is a unique (invariant) compact set $A^* \subseteq X$ such that* $S(A^*) = \bigcup_{i=1}^{n} H_i(A^*) = A^*$;

(ii) *for any compact set A_0 such that $S(A_0) \subseteq A_0$, denoting $A_t = S^t(A_0)$ for $t \geq 1$, we have $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A^*$; furthermore each A_t is compact, and*:

$$d(A_t, A^*) \to 0 \quad \text{as } t \to \infty$$

where d is the Hausdorff distance between compact sets in X.

(iii) *A^* is the support of the unique (invariant) probability distribution, $\mu^* \in \mathcal{P}(X)$, satisfying*

$$\mu^*(B) = \sum_{i=1}^{n} p_i \mu^*[H_i^{-1}(B)] \quad \text{for all } B \in \mathcal{B}(X);$$

(iv) *for $\mu \in \mathcal{P}(X)$, denoting $\mu_t = M^t(\mu)$ for $t \geq 1$:*
   (a) *the support of $\mu_t = S^t$(support of μ),*
   (b) *μ_t converges weakly to μ^*.*

Proposition 4 certainly holds for the IFS $\{G_1, \ldots, G_n; p_1, \ldots, p_n\}$ on Y whenever the Lipschitz constant L defined in (22) of Theorem 1 is such that $L < (1/r_n)$. We shall call A^* the *attractor* of the IFS $\{G_1, \ldots, G_n; p_1, \ldots, p_n\}$ on Y.

For the IFS $\{G_1, \ldots, G_n; p_1, \ldots, p_n\}$, A^* is thus the support of the invariant distribution μ^* to which the distribution of optimal output in the one-sector growth model (discussed in the previous section) converges asymptotically.

4. Application to Cantor type attractors

The principal difficulty in obtaining the invariant distribution of output (using Proposition 4) is that the stochastic process for the output is generated by the functions G_i which are derived from optimal consumption and investment functions, and which one cannot typically solve for in terms of our primitives.

However, the qualitative properties of the G_i (established in Section 3), which *can* be checked given the primitives, allow one to make some qualitative statements about the invariant distribution, depending on the primitives of our model. In this section, we illustrate this by focusing on the case in which there are just two possible outcomes, r_1, r_2, of the random shock, r_1 occurring with probability $p_1 = p$ and r_2 occurring with probability $p_2 = 1 - p$, and demonstrate the possibility of generating invariant distributions which are (a) of the Cantor type, and (b) singular with respect to Lebesgue measure.

The latter illustrates that the support of the invariant distribution can be a "small" subset of[5] $[a, b]$ (more precisely, can be a set of Lebesgue measure zero). In particular, the invariant distribution generated by our model need not be absolutely continuous (that is, need not have a density).

The former illustrates that the support set need not even be a connected set in $[a, b]$. There can be *subintervals* of $[a, b]$, which are not included in the support of the invariant distribution.

Feature (a) depends on the G_i being contractions, and on a property of the IFS, which we call the *no-overlap property*. This is discussed in the first subsection.

Feature (b) depends on the G_i being "sufficiently contracting" in a sense that can be made precise. This is discussed in the second subsection.

4.1. The no-overlap property and contractions

It is well known that if $X = [0, 1]$ and the maps H_1 and H_2 of an IFS of the type described in Section 3.3 are linear with slope ζ, where $0 < \zeta < 1/2$, the attractor A^* of the IFS is a "*middle-α*" Cantor set, where $\alpha = 1 - 2\zeta$ (see, e.g., Mitra et al., 2004).

The maps of the IFS $\{G_1, G_2; p, 1 - p\}$ characterizing the model discussed in the previous sections are clearly nonlinear. The natural question that arises is thus under what conditions such an IFS has an attractor that resembles the typical features of a Cantor type set.

We say that the two maps G_1 and G_2 satisfy the *no-overlap property* if:

$$G_1(b) < G_2(a), \tag{28}$$

so that the maximum of the G_1 function is less than the minimum of the G_2 function on the interval $[a, b]$. An example of maps satisfying the no-overlap property is reported in Fig. 1.

The following result, whose proof can be found in Mitra and Privileggi (2006), provides conditions on the primitives of the model, specifically, $\{r_1, r_2\}$, p, h, δ and k, which ensure the no-overlap property.

Proposition 5. *Suppose the following condition is satisfied*:

$$\frac{\theta}{k} \geq \left(\frac{r_1}{r_2}\right)^2 \tag{29}$$

where θ is defined in (7). *Then the IFS $\{G_1, G_2; p, 1 - p\}$ on $Y = [\theta, k]$ has the no-overlap property* (28).

In view of Section 3, we are now in a position to fully characterize situations where the attractor A^* is a generalized Cantor set directly in terms of the primitives of the model. Proposition 5 gives a sufficient condition for the no-overlap property to hold, and Theorem 1 provides a tool to establish when the maps G_1, G_2 are contractions entirely in terms of the parameters of the model; that is, whenever the constant L defined in (22) is less than $(1/r_2)$.

We shall adopt a general definition of Cantor set based on topological properties. Recall that a set $E \subseteq X$, where (X, d) is a metric space, is said to be *totally disconnected* if its only connected subsets are one-point sets: formally, for any two distinct points x, y in E, there are two non-empty open disjoint sets U and V such that $x \in U$, $y \in V$ and $(U \cap E) \cup (V \cap E) = E$; also, a set $E \subseteq X$ is said to be *perfect* if it is equal to the set of its accumulation points; that is, it is a closed set which contains no isolated points.

Definition 1. We shall say that a set $C \subset \mathbb{R}$ is a *generalized* (*topological*) *Cantor set* on the real line if it is totally disconnected and perfect.

---

[5] Recall that at the end of Section 2 we have defined $a = G_1(a)$ as the largest fixed point of G_1, and by $b = G_n(b)$ the smallest fixed point of G_n; clearly, $X = [a, b] \subset [\theta, k] = Y$. Even if the support of the invariant distribution is either $[a, b]$ itself or a proper subset of $[a, b]$, consistently with Section 3 in the sequel we shall continue to consider our IFS as defined on the larger invariant set $Y = [\theta, k]$.
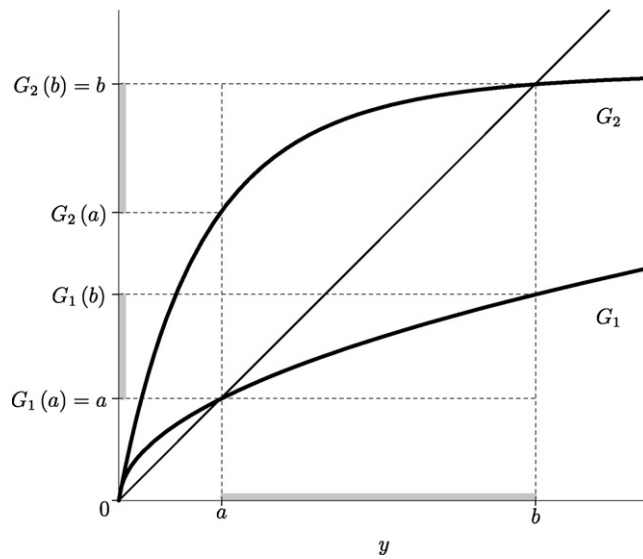
Fig. 1. An example of functions G_1 and G_2 satisfying the no overlap property.

This definition is fully justified in view of the result in Hocking and Young (1961, Chapter 2), where it is established that any compact metric space that is totally disconnected and perfect is homeomorphic to the classical "middle-third" Cantor set.

The next result concludes the work started by Mitra et al. (2004) and successive generalizations in Mitra and Privileggi (2004, 2006) by establishing sufficient conditions entirely in terms of the parameters of the model for the stochastic one-sector optimal growth framework of Brock and Mirman (1972) type to converge to an invariant distribution supported on a generalized Cantor set. It follows directly by applying Theorem 3 in Mitra and Privileggi (2006), and Proposition 5 above.

Proposition 6. *If $L < (1/r_2)$, where L is the constant defined in (22) of Theorem 1, and condition (29) of Proposition 5 holds, then the attractor A^* of the IFS $\{G_1, G_2; p, 1 - p\}$ on $Y = [\theta, k]$, associated to the stochastic process (5), is a generalized (topological) Cantor set.*

Remark 3.

 (i) For the Cobb-Douglas form of the production function, it has been already noticed [see Remark 1 (iii) in Mitra and Privileggi, 2004] that if the exponent of the production function is sufficiently close to zero, then condition (29) and therefore the no-overlap property (28) always holds. In other words, a Cobb-Douglas production function with a low capital coefficient is capable of producing an optimal policy which generates an IFS $\{G_1, G_2; p, 1 - p\}$ with maps that do not overlap.

   It is of interest that a low capital coefficient, which ensures that condition (16) holds (and thereby ensures a positive lower bound on the propensity to consume), as indicated in Remark 1, also ensures that (28) holds (and thereby ensures that the no-overlap property holds).

   If we combine such a production function with a utility function of the iso-elastic type with exponent $(1 - \beta)$, then, as indicated in Remark 2, a low value of β will ensure that $L < (1/r_2)$, so that each G_i is a contraction on $Y = [\theta, k]$.

   Thus, we may conclude that, at least for the Cobb-Douglas production plus iso-elastic utility case, a representative agent with low risk-aversion, operating in an economy with a low capital share in income, will generate an invariant distribution of income supported on a generalized Cantor set.
(ii) It is also worth noting that, whenever G_1 and G_2 are contractions, that is, when $L < (1/r_2)$, then each G_i ($i = 1, 2$) has a unique fixed point: a for G_1 and b for G_2.

4.2. Singular invariant distributions

When the set A^* is of the Cantor type, the question arises as to whether this Cantor-type set has zero Lebesgue measure. This question is non-vacuous since Cantor type sets can have positive Lebesgue measure. They are known as "fat Cantor sets" or Smith-Volterra-Cantor sets. For construction of such sets, see for instance Example 14b in Royden (1968), p. 63, or the example on p. 88 in Hocking and Young (1961).

An important implication of having an attractor of zero Lebesgue measure is that the invariant probability distribution supported on it [see Proposition 4 (iii) and (iv)] is singular. In the following result, we provide a sufficient condition under which the attractor A^* of a IFS with two maps, $\{H_1, H_2; p, 1 - p\}$, has zero Lebesgue measure.

Proposition 7. *Suppose that the maps $H_i : X \to X$, $i \in \{1, 2\}$ are strictly monotone on some closed interval X and constants ℓ_i exist such that $0 < \ell_i < 1$ and $|H_i(y) - H_i(z)| \le \ell_i |y - z|$ for all $y, z \in X$ and for $i \in \{1, 2\}$. If the Lipschitz constants ℓ_i satisfy*

$$\ell_1 + \ell_2 < 1 \tag{30}$$

then the support A^* of the unique invariant distribution μ of the stochastic process (27) is a set of Lebesgue measure zero, and μ is singular with respect to Lebesgue measure.

The proof is reported in Appendix A. In the context of the IFS $\{G_1, G_2; p, 1 - p\}$ on $Y = [\theta, k]$, the property (30) ensures that the no-overlap property (28) also holds. To see this, note that

$$
\begin{aligned}
G_1(b) - G_2(a) &= G_1(b) - G_1(a) + G_1(a) + G_2(b) - G_2(a) - G_2(b) \\
&\le (l_1 + l_2)(b - a) + G_1(a) - G_2(b) \\
&< (b - a) + (a - b) \\
&= 0
\end{aligned}
$$

So the following Corollary is immediate from Theorem 3 in Mitra and Privileggi (2006), and Proposition 7.

Corollary 1. *If $(r_1 + r_2)L < 1$, where L is the constant defined in (22) of Theorem 1, then the support A^* of the unique invariant distribution μ of the IFS $\{G_1, G_2; p, 1 - p\}$ on $Y = [\theta, k]$, associated to the stochastic process (5) is a generalized Cantor set of Lebesgue measure zero, and μ is singular with respect to Lebesgue measure.*

5. Concluding remarks

The results of this work fill a gap left by Mitra and Privileggi (2006) in the study of the geometric properties of the support of the invariant distribution in one-sector stochastic optimal growth models. In this paper, Lipschitz continuity of the iterated function system, generated by the optimal consumption and investment policies, is established directly (in Theorem 1) in terms of the primitives of the model introduced in Section 2. Whenever the parameters of the model are such that the maps of the IFS are contractions, we are able to apply a general result (Proposition 4) on the support of the unique invariant distribution, which is the attractor of the IFS.

In the special case where the random shock takes one of only two values, we provide additional conditions on the parameters of the model which ensure that the IFS satisfies a no-overlap property, and the invariant distribution is supported on a Cantor type set (Proposition 6). A rough interpretation of Proposition 6 is that a low risk-averse representative agent, operating in an economy with a low capital share in income, can generate an invariant distribution of income supported on a generalized Cantor set.

In the special case mentioned above, we provide a sufficient condition on the primitives of the model so that the Cantor type attractor has zero Lebesgue measure and therefore the invariant distribution is singular. The sufficient condition allows us to quantify the Hausdorff dimension of the attractor, and this is our key tool for the result. If the IFS is smooth, alternative approaches might be used to obtain the result, leading possibly to other sufficient conditions. This is suggested by the work of Matsumoto (1988) and Bedford (1991). However, in general, it can be difficult to specify conditions on the primitives leading to smoothness of the IFS (see, e.g., Araujo, 1991; Santos, 1991; Santos and Vigo-Aguiar, 1998). This topic is the matter of future investigation.

Appendix A

Proof of Lemma 1. By using the stochastic Ramsey–Euler Eq. (4) [see Proposition 1(vi)] at $y = a$, we have:

$$
\begin{aligned}
u'[g(a)] &= \delta \mathbb{E}\{u'[g(rG(a))]rh'[\gamma(a)]\} \\
&\le \delta u'[g(r_1 G(a))]h'[\gamma(a)]\mathbb{E}(r) \\
&= \delta u'[g(a)]h'[\gamma(a)]\mathbb{E}(r),
\end{aligned}
$$

where the inequality (in the second line) holds since, for $r \ge r_1$, we have $rG(a) \ge r_1 G(a) \Leftrightarrow g[rG(a)] \ge g[r_1 G(a)] \Leftrightarrow u'\{g[rG(a)]\} \le u'\{g[r_1 G(a)]\}$, and the equality (in the third line) holds since $r_1 G(a) = a$. Thus we get

$$1 \le \delta h'[\gamma(a)]\mathbb{E}(r),$$

which [recall that $(h')^{-1}(\cdot)$ is a decreasing function] yields the left inequality in (14).

Similarly, by using the stochastic Ramsey–Euler Eq. (4) at $y = b$, we have:

$$
\begin{aligned}
u'[g(b)] &= \delta \mathbb{E}\{u'[g(rG(b))]rh'[\gamma(b)]\} \\
&\ge \delta u'[g(r_n G(b))]h'[\gamma(b)]\mathbb{E}(r) \\
&= \delta u'[g(b)]h'[\gamma(b)]\mathbb{E}(r),
\end{aligned}
$$

where the inequality (in the second line) holds since, for $r \leq r_n$, we have $rG(b) \leq r_n G(b) \Leftrightarrow g[rG(b)] \leq g[r_n G(b)] \Leftrightarrow u'\{g[rG(b)]\} \geq u'\{g[r_n G(b)]\}$, and the equality (in the third line) holds since $r_n G(b) = b$. Thus we get:

$$1 \geq \delta h'[\gamma(b)]\mathbb{E}(r),$$

which yields the right inequality in (14). $\quad\square$

Proof of Proposition 3. Since $\gamma(\theta) = \theta - g(\theta)$, (14) and (15) imply:

$$\theta - g(\theta) < \sigma \leq \gamma(b).$$

From the left inequality we get

$$g(\theta) > \theta - \sigma. \tag{31}$$

Since $b = r_n G(b) = r_n h[\gamma(b)]$, we have $\gamma(b) = h^{-1}(b/r_n)$, and thus the right inequality in (14) yields:

$$h^{-1}\left(\frac{b}{r_n}\right) \geq \sigma,$$

from which (since h is increasing) we get

$$\frac{b}{r_n} \geq h(\sigma),$$

which, as $b \leq k$, implies

$$k \geq r_n h(\sigma). \tag{32}$$

Summing up, we have, for all $y \in Y = [\theta, k]$,

$$\frac{g(y)}{y} \geq \frac{g(\theta)}{k} > \frac{\theta}{k} - \frac{\sigma}{k} \geq \frac{\theta}{k} - \frac{\sigma}{r_n h(\sigma)},$$

where the second inequality uses (31) and the last inequality uses (32). $\quad\square$

Proof of Theorem 1. We will first prove that G is *locally Lipschitz* on Y, with Lipschitz constant L. That is, we will show that there is some $\varepsilon > 0$, such that, whenever $y, z \in Y$, and $0 < |y - z| \leq \varepsilon$, (23) holds with L defined by (22).

Denote by m' the minimum value of $[g(y)/y]$ on Y; this is well defined by continuity of $[g(y)/y]$ on the (non-empty) compact set Y. By Proposition 3, we have $m' > m$. Now, define

$$\lambda = \left[1 - \frac{\varepsilon}{\gamma(\theta)}\right]^2 \tag{33}$$

and choose $\varepsilon > 0$ sufficiently small so that

$$\lambda m' > m. \tag{34}$$

It is sufficient to show that, with this choice of ε, whenever $y, z \in Y$ and $0 < z - y \leq \varepsilon$, the inequality (23) holds.

So, let us pick arbitrary $y, z \in Y$ with $0 < z - y \leq \varepsilon$. Now, write Eq. (3) [see Proposition 1(v)] at y and at z:

$$u'[g(y)] = \delta\mathbb{E}\{V'[rG(y)]rh'[\gamma(y)]\}, \tag{35}$$

$$u'[g(z)] = \delta\mathbb{E}\{V'[rG(z)]rh'[\gamma(z)]\}. \tag{36}$$

Subtracting (36) from (35), we obtain:

$$u'[g(y)] - u'[g(z)] = \delta\mathbb{E}\{V'[rG(y)]rh'[\gamma(y)] - V'[rG(z)]rh'[\gamma(z)]\}.$$

Since $rG(z) > rG(y)$, we have $V'[rG(z)] \leq V'[rG(y)]$, and this yields:

$$u'[g(y)] - u'[g(z)] \geq \{h'[\gamma(y)] - h'[\gamma(z)]\}\delta\mathbb{E}\{V'[rG(y)]r\}. \tag{37}$$

We use the mean value theorem to obtain ξ satisfying $g(y) \leq \xi \leq g(z)$, such that:

$$u'[g(y)] - u'[g(z)] = u''(\xi)[g(y) - g(z)]. \tag{38}$$

Similarly, we can find ζ satisfying $\gamma(y) \leq \zeta \leq \gamma(z)$, such that:

$$h'[\gamma(y)] - h'[\gamma(z)] = h''(\zeta)[\gamma(y) - \gamma(z)]. \tag{39}$$

Using (38) and (39) in (37) and changing sign, we obtain:

$$-u''(\xi)[g(z) - g(y)] \geq -h''(\zeta)[\gamma(z) - \gamma(y)]\delta\mathbb{E}\{V'(rG(y))r\}.$$

Dividing through by $u'[g(y)]$ and using (35), the last inequality becomes:

$$-\frac{u''(\xi)}{u'[g(y)]}[g(z) - g(y)] \geq -\frac{h''(\zeta)}{h'[\gamma(y)]}[\gamma(z) - \gamma(y)],$$

which, since $\gamma(z) = z - g(z)$ and $\gamma(y) = y - g(y)$, may be rewritten as

$$-\frac{u''(\xi)}{u'[g(y)]}[g(z) - g(y)] \geq -\frac{h''(\zeta)}{h'[\gamma(y)]}\{(z - y) - [g(z) - g(y)]\}.$$

Rearranging terms, we get

$$\frac{g(z) - g(y)}{z - y} \geq -\frac{h''(\zeta)}{h'[\gamma(y)]}\left\{-\frac{u''(\xi)}{u'[g(y)]} - \frac{h''(\zeta)}{h'[\gamma(y)]}\right\}^{-1}. \tag{40}$$

It remains to convert the right-hand side into terms involving the parameters of our model.
Note that since ξ satisfies $g(y) \leq \xi$, we have:

$$-u''(\xi) \leq \frac{B'u'(\xi)}{\xi} \leq \frac{B'u'[g(y)]}{g(y)} \leq \left(\frac{B'}{B}\right)\{-u''[g(y)]\},$$

so that:

$$-\frac{u''(\xi)}{u'[g(y)]} \leq \left(\frac{B'}{B}\right)\left\{-\frac{u''[g(y)]g(y)}{u'[g(y)]}\right\}\frac{1}{g(y)} \leq \frac{(B')^2}{Bg(y)} = \frac{\beta'}{g(y)}. \tag{41}$$

Since ζ satisfies $\zeta \leq \gamma(z)$, we have:

$$-h''(\zeta) \geq \frac{Ah'(\zeta)}{\zeta} \geq \frac{Ah'[\gamma(z)]}{\gamma(z)} \geq \left(\frac{A}{A'}\right)\{-h''[\gamma(z)]\},$$

so that:

$$\begin{aligned}
\frac{-h''(\zeta)}{h'[\gamma(y)]} &\geq \left(\frac{A}{A'}\right)\left\{-\frac{h''[\gamma(z)]}{h'[\gamma(z)]}\right\}\left\{\frac{h'[\gamma(z)]}{h'[\gamma(y)]}\right\} \\
&= \left(\frac{A}{A'}\right)\left\{-\frac{h''[\gamma(z)]}{h'[\gamma(z)]}\right\}\left[\frac{\gamma(z)}{\gamma(z)}\right]\left\{\frac{h'[\gamma(z)]}{h'[\gamma(y)]}\right\} \\
&\geq \left(\frac{A^2}{A'}\right)\left[\frac{1}{\gamma(z)}\right]\left\{\frac{h'[\gamma(z)]}{h'[\gamma(y)]}\right\} \\
&= \left(\frac{A^2}{A'}\right)\left[\frac{\gamma(y)}{\gamma(z)}\right]^2\left\{\frac{h'[\gamma(z)][\gamma(z)]}{h'[\gamma(y)][\gamma(y)]}\right\}\left[\frac{1}{\gamma(y)}\right] \\
&\geq \left(\frac{A^2}{A'}\right)\left[\frac{\gamma(y)}{\gamma(z)}\right]^2\left\{\frac{A''h[\gamma(z)]}{h[\gamma(y)]}\right\}\left[\frac{1}{\gamma(y)}\right] \\
&\geq \alpha'\left[\frac{\gamma(y)}{\gamma(z)}\right]^2\left[\frac{1}{\gamma(y)}\right] \\
&> \alpha\left[1 - \frac{\varepsilon}{\gamma(\theta)}\right]^2\left[\frac{1}{\gamma(y)}\right],
\end{aligned}$$

where the last but one inequality holds thanks to (20) in Fact 1 and the last inequality uses the assumption $z - y \leq \varepsilon$, as the following inequalities show:

$$\frac{\gamma(y)}{\gamma(z)} = \frac{\gamma(z) - [\gamma(z) - \gamma(y)]}{\gamma(z)} = 1 - \frac{z - g(z) - [y - g(y)]}{\gamma(z)} > 1 - \frac{z - y}{\gamma(z)} \geq 1 - \frac{\varepsilon}{\gamma(\theta)}.$$

In view of (33), we thus obtain:

$$-\frac{h''(\zeta)}{h'[\gamma(y)]} > \frac{\alpha'\lambda}{\gamma(y)}. \tag{42}$$

Using (41) and (42) in (40) and rearranging terms, we get:

$$\frac{g(z) - g(y)}{z - y} > \alpha'\left[\frac{\lambda g(y)}{y}\right]\left\{\beta'\left[1 - \frac{g(y)}{y}\right] + \alpha'\left[\frac{\lambda g(y)}{y}\right]\right\}^{-1}$$

which boils down to

$$\frac{g(z) - g(y)}{z - y} > \frac{\alpha' m}{\beta'\{1 - [g(y)/y]\} + \alpha' m} > \frac{\alpha' m}{\beta'(1 - m) + \alpha' m}, \tag{43}$$

where the first inequality holds since, by definition of m' and by (34), $[\lambda g(y)/y] \geq \lambda m' > m$; similarly, the second one follows from $[g(y)/y] \geq m' > m$.

By definition of G, we have:

$$
\begin{aligned}
\frac{G(z) - G(y)}{z - y} &= \frac{h[\gamma(z)] - h[\gamma(y)]}{z - y} \\
&\leq \frac{h'[\gamma(y)][\gamma(z) - \gamma(y)]}{z - y} \\
&= h'[\gamma(y)] \frac{(z - y) - [g(z) - g(y)]}{z - y} \\
&\leq h'[\gamma(\theta)] \left[1 - \frac{g(z) - g(y)}{z - y} \right] \\
&\leq h'[\gamma(\theta)] \left[1 - \frac{\alpha' m}{\beta'(1 - m) + \alpha' m} \right] \\
&= h'[\gamma(\theta)] \left[\frac{\beta'(1 - m)}{\beta'(1 - m) + \alpha' m} \right] \\
&\leq \frac{\beta'(1 - m)}{\delta p_1 r_1 [\beta'(1 - m) + \alpha' m]} = L,
\end{aligned}
$$

where in the fifth line we used (43) and the last line follows from condition (11) in Proposition 2. This establishes that G is locally Lipschitz on Y, with Lipschitz constant L.

It follows from the above result that G is Lipschitz continuous on Y with Lipschitz constant L. To see this, pick any $y', z' \in Y$ with $0 < z' - y'$. We can find a positive integer n, such that $n\varepsilon \geq (z' - y')$, where ε was used in the definition of (33). Define $\eta = (z' - y')/n$; then $0 < \eta \leq \varepsilon$. We use η to define:

$$(y_0, y_1, \ldots, y_n) = (y', y' + \eta, \ldots, y' + (n - 1)\eta, z')$$

Then, we have, using the fact that G is locally Lipschitz [with the choice of ε used in the definition of (33)] with Lipschitz constant L, and $0 < (y_{j+1} - y_j) = \eta \leq \varepsilon$ for $j = 0, 1, \ldots, n - 1$,

$$G(z') - G(y') = \sum_{j=0}^{n-1} [G(y_{j+1}) - G(y_j)] \leq L \sum_{j=0}^{n-1} (y_{j+1} - y_j) = L(z' - y').$$

This establishes that G is Lipschitz continuous on Y, with Lipschitz constant L. □

Proof of Proposition 7. By Proposition 9.6 in Falconer (2003), p. 135, the Hausdorff dimension of A^* cannot exceed the (unique) positive root $\tilde{d}$ of the equation:

$$\ell_0^d + \ell_1^d = 1.$$

Given (30), we must have $\tilde{d} < 1$. Thus, the the Hausdorff dimension of A^* is less than 1. By definition of Hausdorff dimension (see Chapter 2 in Falconer, 2003), the Hausdorff outer measure of A^* is zero. Since Lebesgue outer measure coincides with Hausdorff outer measure on $\mathbb{R}$, the Lebesgue outer measure of A^* is zero. Since A^* is closed, it is measurable, and hence the Lebesgue measure of A^* is zero.

Since A^* is the support of μ, by definition of support (see, e.g., Chung, 1974, p. 31) $\mu(X - A^*) = 0$. We have just seen that $\nu(A^*) = 0$, where ν is Lebesgue measure; thus, μ is singular with respect to Lebesgue measure (see Billingsley, 1979, p. 374). □

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